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# Connected Sets in Global Bifurcation Theory

John Toland (Bath)
joint work with Boris Buffoni (Lausanne)

## Abstract

The ideas in this lecture will be illustrated by, but are not restricted to, the problem

$$x = \lambda Lx + R(\lambda, x),\ \lambda \in \mathbb{R},\ x \in X \text{ a real Banach space,} \tag{‡}$$

where $L : X \to X$ is compact and linear, $R : \mathbb{R} \times X \to X$ is compact and continuous, $R(\lambda, 0) = 0,\ \lambda \in \mathbb{R}$, and $\|R(\lambda, x)\| = \circ(\|x\|)$ as $\|x\| \to 0$, locally uniformly in $\lambda$.

Under these modest hypotheses it is known that global connected sets of non-trivial solutions bifurcate from the trivial solution $(\lambda_0, 0)$ when $\lambda_0$ is a characteristic value[1] of odd algebraic multiplicity of $L$. Related results are that

- these global sets are path-connected if $R$ is real-analytic
- there are simple examples with infinitely differentiable $R$ for which the only path-connected components of the connected sets that bifurcate are singletons.

Based on elementary but subtle results in the point-set topology of locally compact, connected, metric spaces (which will be explained) the lecture will sketch proofs that these global sets of solutions are path-connected when either

- $R : \mathbb{R} \times X \to X$ is real-analytic except at countably many points in $\mathbb{R} \times X$

or

- for each $\lambda \in \mathbb{R}$ the solutions $x \in X$ of (‡) are isolated. □

A MORE GENERAL CRITERION FOR PATH-CONNECTEDNESS
WHICH IS DIFFICULT TO DESCRIBE IN TERMS OF PROPERTIES OF (‡)
LEADS TO A CHALLENGING OPEN QUESTION

[1] $\lambda_0$ is a characteristic value of $L$ if $\lambda_0 \neq 0$ and $1/\lambda_0$ is an eigenvalue of $L$.